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Analysis of unsteady waves in solids*

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In the present paper the acceleration wave theory applicable to the analysis of unsteady waves is developed. It is noted that a measurement of either the stress or the velocity history at a material point is sufficient to determine the history of the remaining one of these variables and of the strain if the instantaneous sound speed is known. This sound speed can be approximated by the speed determined as the wave passes through the material between closely spaced gauge stations or may be directly calculated from simultaneous measurements of particle velocity and stress at a single station. The present theory permits the analysis of wave-profile data obtained using conventional instrumentation which has a time resolution of a few nanoseconds.

I. INTRODUCTION

When a solid is subjected to the rapid impulsive loading accompanying impact or explosion, waves propagate with amplitudes and speeds which are governed by the mechanical properties of the sample. After suitable analysis, experimental measurements of the speeds and amplitudes of these waves are frequently utilized to obtain a description of the constitutive relation of the shock-loaded sample. In the earliest investigations, experimental data were obtained only for pressures well above 100 kbar and were analyzed with conservation relations which assumed that compressive waves propagated as steady shocks. At high pressure this assumption is valid to within the limits of available instrumentation to make time-resolved measurements. More recent work in the pressure range of approximately 1—100 kbar has shown that compression waves may be unsteady, frequently exhibit appreciable rise times, and often possess a very complex structure.

While the theoretical and experimental bases for high-pressure shock-wave studies are well established and widely employed,[1–3] analysis of experiments in which recorded waveforms exhibit finite rise times is more complicated and less certain. In the simplest instance, these structured waves consist simply of two or more shocks and can be interpreted in terms of the shock jump conditions with little difficulty. Observed waveforms involving smooth transitions between states have been analyzed by approximating the observed stress or particle-velocity history as a sequence of shocks so that the approximate waveform can be analyzed by means of the jump conditions for shocks.[4–6] Even though this procedure for treating unsteady waveforms seems to provide a suitable data-reduction technique in most instances, the extent of the approximations has not been defined and no explicit theoretical basis for the analysis has been offered. A well-founded and effective scheme for reducing the data obtained in conventional shock-compression experiments conducted in the low-pressure regime is urgently needed. Since there is no universally acceptable constitutive equation for describing the nonequilibrium behavior of most of the various classes of materials that propagate waves exhibiting appreciable structure, it is desirable that the data-reduction scheme be one in which information can be extracted without the assumption of any such relation.

A recent examination of this problem by Fowles and Williams[7] has given rise to a new theory of unsteady wave propagation. This theory involves different speeds of propagation of stress and particle-velocity waves; hence, it will be referred to as the dual-wave theory. Unfortunately, attempts to base a data-reduction scheme on the dual-wave theory have been frustrated by the necessity to employ instrumentation of a type that, in its present state of development, does not exhibit sufficiently good time resolution to study many materials of interest.

In this paper an alternative scheme is presented which, while not directly meeting the challenge posed by the dual-wave theory, has the advantage of providing a rational and explicit method for reducing data obtained from conventional instrumentation. This method is similar to that in which the wave is approximated as a sequence of steps (shocks) but employs the higher approximation in which the experimental record is approximated as a sequence of chords. The discontinuities of slope where the chords join form acceleration waves, and the data-reduction method proposed in this paper is based on the theory of these waves. Only a single wave speed (the acceleration wave speed) appears in the governing equations, which are otherwise similar to those of the dual-wave theory. The proposed data-reduction scheme occupies an intermediate position between the conventional method of approximating the record as a sequence of shocks and the dual-wave method of Fowles and Williams. It is better founded than the former method, and yet still preserves its essential advantages of experimental simplicity and ability to cope with wave interactions.

After considering kinematical preliminaries in Sec. II and pertinent results from the theory of singular surfaces in Sec. III, the properties of acceleration waves will be considered in Sec. IV. In Sec. V, the analysis will be specialized to the case of one-dimensional motions, and in the discussion of Sec. VI the acceleration wave theory will be compared to other results. In the Appendix it is shown that the analysis can be employed to solve wave interaction problems.

II. KINEMATICAL PRELIMINARIES

Let $\mathbf{X}=(X^1, X^2, X^3)$ denote the coordinates of a material point at time t_0, and let $\mathbf{x}=(x^1,x^2,x^3)$ denote the coordinates of the same material point at time $t \geq t_0$. A motion of a material body is described by the function $\hat{\mathbf{x}}$ such that

$$\mathbf{x}=\hat{\mathbf{x}}(\mathbf{X}, t), \tag{2.1}$$

with the property $\hat{\mathbf{x}}(X, t_0)=\mathbf{X}$. The physical interpretation of the function $\hat{\mathbf{x}}$ is that it gives the coordinates $\mathbf{x}$ of a material point at each time $t > t_0$, whose position at $t=t_0$ is given by the coordinates $\mathbf{X}$. In the treatment of physical problems, the usage of either $(\mathbf{X}, t)$ or $(\mathbf{x}, t)$ as independent variables is equivalent. $(\mathbf{X}, t)$ is referred to as the material description while $(\mathbf{x}, t)$ is referred to as the spatial description. In this paper we employ the material description, because we have in mind application to problems of solid mechanics in which the instrumentation is affixed to a specific particle and records the history of events taking place at this particle.

The components $\dot{x}^k$ of the velocity $\dot{\mathbf{x}}$ of a particle and the components $\ddot{x}^k$ of the acceleration $\ddot{\mathbf{x}}$ of a particle are given by

$$\dot{x}^k=\frac{\partial \hat{x}^k(\mathbf{X},t)}{\partial t}, \quad \ddot{x}^k=\frac{\partial^2 \hat{x}^k(\mathbf{X},t)}{\partial t^2}. \tag{2.2}$$

At each $t > t_0$, the components $x^k_{,K}$, $x^k_{,KL}$ of the deformation gradient and its derivative, respectively, are given by

$$x^k_{,K}=\frac{\partial \hat{x}^k(\mathbf{X},t)}{\partial X^K} \quad \text{and} \quad x^k_{,KL}=\frac{\partial \hat{x}^k(\mathbf{X},t)}{\partial X^K\,\partial X^L}. \tag{2.3}$$

III. RESULTS FROM THE THEORY OF SINGULAR SURFACES

The theory of singular surfaces and the conditions that must be satisfied across a singular surface for geometrical and for kinematical reasons have been presented, for instance, by Truesdell and Toupin.[8] Here only the main features of singular surface analysis will be reviewed and applied to the problem at hand.

Consider a material region R and a surface S that divides the region into R^+ and R^-. The unit normal $\mathbf{N}$ to the surface S is directed toward R^+. Let $\Psi(\cdot, t)$ be a function which is continuous within the regions R^+ and R^- at each $t > t_0$ and for which the limits Ψ^+ and Ψ^- exist as $\mathbf{X}$ approaches a point $\mathbf{X}_0$ on S along paths wholly within R^+ and R^-. We say that the surface S is singular with respect to $\Psi(\mathbf{X}, t)$ if

$$[\Psi] \equiv \Psi^+ - \Psi^- \neq 0.$$

If the surface S is a moving surface, i.e., a wave, then it is necessary to discuss the manner in which it may move. Basically, there are three ways of describing speeds by which this can be accomplished. (i) u_n—the speed of displacement. It is a measure of the speed with which the surface moves with respect to the origin of our fixed rectangular Cartesian system. (ii) U_N—the speed of propagation. It is a measure of the speed with which the surface traverses the material. (iii) U—the local speed of propagation. It is a measure of the speed with which the surface moves with respect to the particles instantaneously on the surface, i.e., $U=u_n-\dot{x}_n$. $\dot{x}_n$ is the normal component of the velocity of the particles instantaneously on the surface with respect to the spatial direction $\mathbf{n}$ in which the surface is moving.

In the present discussion, the following definitions are needed:

(i) A wave is said to be a shock wave if the conditions

$$[x^k]=0, \quad [\dot{x}^k] \neq 0, \quad \text{and} \quad [x^k_{,K}] \neq 0 \tag{3.1}$$

are satisfied.

(ii) A wave is said to be an acceleration wave if

$$[x^k]=[\dot{x}^k]=[x^k_{,K}]=0, \quad [\ddot{x}^k] \neq 0, \quad [\dot{x}^k_{,K}] \neq 0, \quad \text{and} \quad [x^k_{,KM}] \neq 0. \tag{3.2}$$

Higher-order waves can be defined in an analogous manner.

A singular surface is called a contact surface if $[\dot{\mathbf{x}}]=0$ and $U=0$. A contact surface has no motion with respect to the material but may, of course, be convected along with the material.

The requirements of conservation of mass and conservation of linear momentum across a singular surface are of the form

$$[\rho U]=0, \tag{3.3}$$

and

$$[t^{km}]n_m + [\rho U \dot{x}^k]=0, \tag{3.4}$$

where t^{km} is called the Cauchy stress or the true stress, and ρ is the present density.

It should be pointed out that relations (3.3) and (3.4) are arrived at independent of the nature of the surface. These are the relations which must be satisfied for every wave or contact surface if mass and linear momentum are to be conserved across the surface. For the case of shock waves when values of U and $[\dot{x}^k]$ are known, these relations are used to determine the value of the density and the value of the stress components behind the shocks. Furthermore, these relations are arrived at independent of the material in question, i.e., independent of the constitutive relations.

The foregoing equations are valid locally and instantaneously, i.e., at a given particle X and time t, no matter what the conditions may be at surrounding particles or previous times. The conditions need not be, and usually are not, uniformly valid in the sense that any quantity appearing in them remains constant. Numerous discussions of the growth or decay of waves have been given.[9,10]

IV. ACCELERATION WAVES

In this section, attention will be directed to the analysis of acceleration waves. First, the consequences of the conditions of definition of an acceleration wave are examined. Since $U = u_n - \dot{x}_n$ and $[u_n]=0$, then $[U] = -[\dot{x}_n]$. Since $[\dot{x}^k]=0$ for an acceleration wave, it follows that $[\dot{x}_n]=0$ and

$$[U]=0. \tag{4.1}$$

It is immediately clear that by (3.3)

$$[\rho]=0. \tag{4.2}$$

By (4.1), (4.2), and the definition of an acceleration wave, (3.4) takes the form

$$[t^{km}]n_m=0 \tag{4.3}$$

for an acceleration wave. Now, consider the Piola-Kirchhoff stress T^{kK} which is related to the Cauchy stress t^{km} by the equation[11]

$$t^{km}n_m da = T^{kK}N_K dA, \tag{4.4}$$

where da is the area element in the deformed material and dA is the area element in the undeformed material. Here, the relation between n and N is given by Eq. (182.8) of Ref. 8. By (4.3) and (4.4) it follows immediately that

$$[T^k]=[T^{kK}]N_K=0, \tag{4.5}$$

where $T^k = T^{kK}N_K$. Therefore, by requiring relations (3.3) and (3.4) be satisfied, it follows that the density and the stress vector must be continuous across an acceleration wave.

The jumps in the second derivatives of motion are given by

$$\begin{aligned}[\ddot{x}^k] &= U^2 a^k,\\ [\dot{x}^k_{,m}] &= -Ua^k n_m,\end{aligned} \tag{4.6}$$

where a^k are the components of an arbitrary surface vector called the wave amplitude. It follows immediately that

$$[\dot{x}^k_{,K}]U = -[\ddot{x}^k]n_k. \tag{4.7}$$

Since the differential form of the conservation of mass must be satisfied on either side of a singular surface, then

$$[\dot{\rho} + \rho \dot{x}^k_{,K}]=0, \tag{4.8}$$

which reduces to

$$[\dot{\rho}] + \rho[\dot{x}^k_{,K}]=0 \tag{4.9}$$

for an acceleration wave. Furthermore, (4.9) with (4.7) becomes

$$[\dot{\rho}]U - \rho[\ddot{x}^k]n_k=0. \tag{4.10}$$

Since $[T^k]=0$, and applying the dual of Eq. (180.4) of Ref. 8 to T^k, it follows that

$$[\dot{T}^k] = -U_N[N^L T^k_{,L}] = -U_N[N^L T^{kK}_{,L}N_K + T^{kK}N^L N_{K,L}]. \tag{4.11}$$

In addition, if $[T^{kK}]=0$, it can be shown that (4.11) together with the differential form of the conservation of linear momentum reduces to

$$[\dot{T}^{kK}]N_K = -\rho_0 U_N[\ddot{x}^k], \tag{4.12}$$

where ρ_0 is the density at $t=t_0$. Furthermore, it can be shown without any additional assumptions that (4.12) has the alternate form

$$[\dot{t}^{km}]n_m = -\rho U[\ddot{x}^k]. \tag{4.13}$$

Thus, two very important relations, given by (4.10) and (4.13), have been derived. It is clear that these conditions are arrived at independent of the material in question. Therefore, they may be used to determine the rate of change of density and the rate of change of the stress behind an acceleration wave. On the other hand, (4.13) may be used to determine the local speed of propagation of an acceleration wave in a particular material when its constitutive relation is given.

V. SPECIALIZATION TO ONE-DIMENSIONAL MOTIONS

The relations (4.10) and (4.13) are, of course, quite general and must be specialized for application to the analysis of plane longitudinal one-dimensional motions. Let the x^1 axis be coincident with such a motion. Then $n^1=1$ and $\ddot{x}^1 \equiv \ddot{x}$ are the only nonzero components of the normal and the acceleration. In this situation (4.10) and (4.13) imply that

$$[\dot{\rho}]U - \rho[\ddot{x}]=0,$$

and (5.1)

$$[\dot{\sigma}] + \rho U[\ddot{x}]=0,$$

where $\sigma \equiv t^{11}$. The strain ϵ is given by

$$\epsilon = (\rho_0/\rho) - 1.$$

Consequently,

$$\dot{\rho} = -(\rho^2/\rho_0)\dot{\epsilon};$$

and since $\rho U = \rho_0 U_N$, the relations (5.1) may be rewritten in the forms

$$\begin{aligned}[\dot{\epsilon}] &= -(1/U_N)[\ddot{x}],\\ [\dot{\sigma}] &= -\rho_0 U_N[\ddot{x}],\end{aligned} \tag{5.2}$$

which are most convenient for the analysis of particle-velocity–time data. When stress-time data are to be reduced, the forms

$$\begin{aligned}[\dot{\epsilon}] &= (1/\rho_0 U_N^2)[\dot{\sigma}],\\ [\ddot{x}] &= -(1/\rho_0 U_N)[\dot{\sigma}]\end{aligned} \tag{5.3}$$

are appropriate. The quantity ρ_0, the density of the body in the reference state, is a constant known in advance. The use of the measure U_N of the wave speed is also

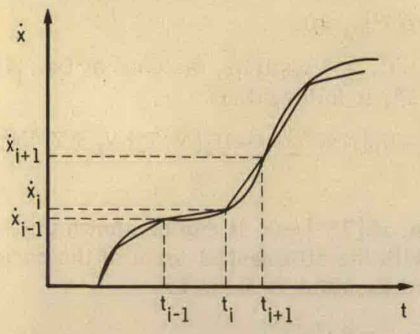

FIG. 1. Recorded velocity history with an approximation by a sequence of chords. The discontinuities where the chords join represent acceleration waves.

convenient, as it is the quantity inferred by measurement of wave transit time between given fixed material particles.

To illustrate the application of Eq. (5.1) to the interpretation of experimental data the particle-velocity history shown in Fig. 1 is considered as an example. This history is considered to have been recorded at a given plane in the interior of a sample without interference with the propagating wave. The actual recorded waveform is approximated by a system of chords such as that shown. Further attention is restricted to this piecewise linear approximation. The discontinuities in slope where the chords meet are acceleration waves, and are treated as outlined in the preceding paragraphs.

If, for simplicity, the chord approximation is chosen so that the arrivals of the acceleration waves at the recording station are all equally spaced in time by an amount Δt, the particle accelerations in the regions ahead of an behind the ith wave are

$$\ddot{x}_i^+ = (\dot{x}_i - \dot{x}_{i-1})/\Delta t, \quad \ddot{x}_i^- = (\dot{x}_{i+1} - \dot{x}_i)/\Delta t,$$

respectively, and we have

$$[\ddot{x}]_i = -(\dot{x}_{i-1} - 2\dot{x}_i + \dot{x}_{i+1})/\Delta t. \qquad (5.4)$$

Similar relations hold for other quantities of interest of interest although they will be only approximately correct, even if the actual particle-velocity record is piecewise linear. Relations corresponding to (5.4) for jumps in stress and strain rates are

$$[\dot{\sigma}]_i = -(\sigma_{i-1} - 2\sigma_i + \sigma_{i+1})/\Delta t,$$
$$[\dot{\epsilon}]_i = -(\epsilon_{i-1} - 2\epsilon_i + \epsilon_{i+1})/\Delta t. \qquad (5.5)$$

Substitution of (5.4) and (5.5) into (5.2) gives the formulas

$$\epsilon_{i+1} = 2\epsilon_i - \epsilon_{i-1} - (1/U_{Ni})(\dot{x}_{i-1} - 2\dot{x}_i + \dot{x}_{i+1}),$$
$$\sigma_{i+1} = 2\sigma_i - \sigma_{i-1} - \rho_0 U_{Ni}(\dot{x}_{i-1} - 2\dot{x}_i + \dot{x}_{i+1}), \qquad (5.6)$$

from which the stress and strain at the $(i+1)$st wave can be calculated from the stress and strain at previous waves, recorded values of particle velocity, and a knowledge of the propagation velocity of the ith wave. A similar calculation based on (5.3) yields formulas for finding ϵ and $\dot{x}$ from measured stress histories.

It should be pointed out that the quantity $U_{Ni}(t)$ appearing in (5.6) is the acceleration wave speed which, for very large classes of materials, is the sound speed.[12,13] In order to carry out data reduction with (5.6), this speed must be known in advance or determined experimentally. First, if the velocity-time data or the stress-time data are available for two stations, then the instantaneous wave speed may be approximated from calculations based on the difference in arrival times of each particle-velocity or stress level. On the other hand, if both the velocity-time data and the stress-time data are available at a single station, then the wave speed follows directly by applying (5.6).

Finally, it should be pointed out that many experiments involve perturbations to the incident wave shapes. As demonstrated in the Appendix, the acceleration wave theory provides a basis for solving the resulting wave interaction problems.

VI. DISCUSSION

The acceleration wave theory presented here and the theory of Fowles and Williams[7] are broadly applicable to many solids because these theories involve no assumptions concerning the constitutive relation. However, the two theories lead to significantly different demands upon the experimental determination of material response from wave-propagation experiments. The dual-wave theory requires measurements of both stress and velocity histories. On the other hand, the acceleration wave analysis requires only measurement of either the stress or velocity history. Both theories require measurements of wave speed. In the acceleration wave theory, the average speed of either the stress or the velocity wave between two closely spaced stations is employed as an approximation to the instantaneous sound speed, whereas the dual-wave theory requires simultaneous measurements of both stress and velocity waves. According to the acceleration wave theory, simultaneous measurements of stress and particle velocity at a single station permits the instantaneous acceleration wave speed to be calculated.

In effect, the acceleration wave theory places no new demands upon the experiments beyond those customary in shock-wave studies, even though the theory is applicable to more complex material response. Since previous analyses of waves are known to provide a close approximation to the real response, the acceleration wave theory should provide an adequate base for analyzing the response of many solids with the use of existing experimental instruments and techniques. Furthermore, as shown in the Appendix, the acceleration wave theory leads to fairly simple and straightforward techniques for solving wave interaction problems which are frequently introduced by the measuring instruments.

In contrast to the acceleration wave theory, the dual-wave theory appears to require experiments which are not possible with existing capabilities. Furthermore, if that capability were developed it is not clear whether it would result in any significant change in the end result of describing material response. Butcher[14] has performed computer analyses of several rate-dependent solids which indicate that differences in wave speeds are too small to be significant.

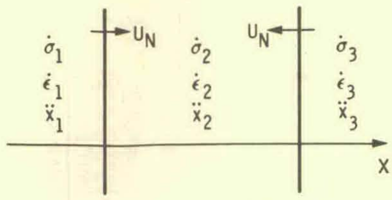

FIG. 2. Acceleration waves at the instant before collision.

In considering experimental capabilities it is important to observe that the theories are only concerned with situations in which significant unsteady behavior is expected. Hence, the interactions between characteristic material relaxation time, characteristic instrumentation time, and wave transit time between measuring stations must be explicitly considered. Most authors who have discussed implementation of the dual-wave theory[15] have implicitly assumed that instruments have the ideal capabilities of time resolution of a few nanoseconds in changes in velocity or stress, coupled with an *in situ* location at which the velocity or stress can be measured without perturbing the disturbance. Except in special cases these capabilities do not exist.

The ideal conditions can be obtained at present if a Sandia velocity interferometer[6] is employed to measure the velocity history of an imbedded mirror in an optically clear solid.[16] (This measurement does not satisfy the requirements of the dual-wave theory since only the velocity history is measured.) A close approximation to the ideal measurement is achieved when a velocity interferometer with a fused quartz window or a Sandia quartz gauge[17] is employed to measure the response of materials such as aluminum or beryllium which have good impedance matches to the gauge. When the velocity interferometer is employed with a sapphire window or a Sandia sapphire gauge[18] is employed to investigate iron, good time resolution and impedance matching are also obtained.

Techniques have been developed to imbed various gauges directly in solid samples. These gauges include a manganin gauge,[19,20] a dual velocity and stress gauge,[21] and an electromagnetic stress gauge.[22] However, experimental studies to date have demonstrated time resolutions of no better than about 50 nsec. Furthermore, the uniqueness of the gauge responses in various solid assemblies remains in question.

If materials with significant relaxation rates are to be investigated, time resolution of velocity or stress of 50 nsec leads to either insufficient accuracy in wave speed and amplitude measurements over short gauge spacings or insufficient accuracies in the amplitude changes between large gauge spacings. A recent paper[23] analyzes the unsteady behavior associated with elastic precursor decay, one of the most unsteady phenomena associated with shock-wave loading. The data analysis presented in that paper can be used as an explicit example of the interaction of unsteady behavior, measurements of rapidly changing amplitudes, and the dual-wave velocities. Although dual-wave speeds were not measured in the paper, reasonable assumptions concerning the constitutive relation were used to compute the speeds of velocity and stress waves. The difference is small (4% in the most unsteady case shown). Even though the difference in wave speed is small, a very large error is introduced if a gauge used to measure the local stress history has inadequate time-response capabilities. For example, data in the paper show that the stress amplitude changes 30% in a 50-nsec interval when the wave speeds differ by 4%. Thus, a gauge limited to a characteristic time of 50 nsec for accurate detection of changes in stress would incorporate errors in amplitude which are much greater than possible differences in wave speeds.

This example serves to emphasize that the most important consideration for the determination of unsteady behavior is the capability to accurately determine changes in amplitude with a time resolution of a few nanoseconds. The acceleration wave theory appears to provide an adequate description for most unsteady wave measurements accomplished with existing gauges.

APPENDIX: INTERACTIONS OF PLANE ACCELERATION WAVES OF UNIAXIAL STRAIN

The foregoing work has dealt with conditions that must be satisfied at isolated acceleration waves. In the following we examine the results of various kinds of acceleration wave interactions. These results on wave interactions are necessary if we are to use the theory in the interpretation of most kinds of experimental results, because it is usually necessary to account for the interaction of waves with each other inside samples, and with gauges or free surfaces at the boundary of samples.

Collision of two acceleration waves

At the instant before collision we have the situation shown in Fig. 2. The field variables in the initial state are constrained by their having to satisfy the jump conditions (5.2), which we write in the form

$$[\dot\epsilon]U_N + [\ddot x] = 0, \quad [\dot\sigma] + \rho_0 U_N [\ddot x] = 0.$$

Applying these conditions to each of the two waves we have

$$(\dot\epsilon_2 - \dot\epsilon_1)U_N + (\ddot x_2 - \ddot x_1) = 0, \quad (\dot\sigma_2 - \dot\sigma_1) + \rho_0 U_N(\ddot x_2 - \ddot x_1) = 0,$$
$$\text{(A1)}$$
$$(\dot\epsilon_3 - \dot\epsilon_2)U_N - (\ddot x_3 - \ddot x_2) = 0, \quad (\dot\sigma_3 - \dot\sigma_2) - \rho_0 U_N(\ddot x_3 - \ddot x_2) = 0.$$

Following the collision the situation is as shown in Fig. 3 and the jump conditions must again be satisfied at the waves, giving

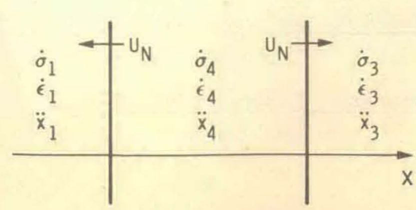

FIG. 3. Acceleration waves at the instant after collision.

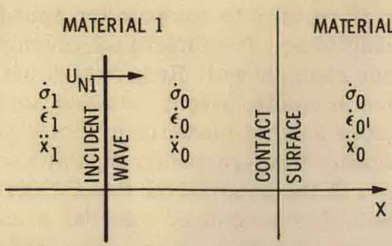

FIG. 4. Acceleration wave incident upon a contact surface.

$$(\dot{\epsilon}_4-\dot{\epsilon}_1)U_N-(\ddot{x}_4-\ddot{x}_1)=0,\quad (\dot{\sigma}_4-\dot{\sigma}_1)-\rho_0U_N(\ddot{x}_4-\ddot{x}_1)=0, \tag{A2}$$

$$(\dot{\epsilon}_3-\dot{\epsilon}_4)U_N-(\ddot{x}_3-\ddot{x}_4)=0,\quad (\dot{\sigma}_3-\dot{\sigma}_4)+\rho_0U_N(\ddot{x}_3-\ddot{x}_4)=0.$$

Solving these equations for $\dot{\sigma}_4$, $\dot{\epsilon}_4$, and $\ddot{x}_4$ we have

$$\begin{aligned}\dot{\sigma}_4&=\tfrac{1}{2}[(\dot{\sigma}_1+\dot{\sigma}_3)-\rho_0U_N(\ddot{x}_1-\ddot{x}_3)],\\ \dot{\epsilon}_4&=\tfrac{1}{2}[(\dot{\epsilon}_1+\dot{\epsilon}_3)-(1/U_N)(\ddot{x}_1-\ddot{x}_3)],\\ \ddot{x}_4&=\tfrac{1}{2}[(\ddot{x}_1+\ddot{x}_3)-(\dot{\epsilon}_1-\dot{\epsilon}_3)U_N].\end{aligned} \tag{A3}$$

Not all the quantities in the right members of these equations are independent; they must satisfy (A1). Using these equations we rewrite (A3) in the equivalent form

$$\dot{\sigma}_4=\dot{\sigma}_1-\dot{\sigma}_2+\dot{\sigma}_3,\quad \dot{\epsilon}_4=\dot{\epsilon}_1-\dot{\epsilon}_2+\dot{\epsilon}_3,\quad \ddot{x}_4=\ddot{x}_1-\ddot{x}_2+\ddot{x}_3. \tag{A4}$$

Interaction of an acceleration wave with a contact surface

At the instant before collision of an acceleration wave with a contact surface we have the situation shown in Fig. 4. In conformity with (5.1) and the definition of a contact surface, we have $\dot{\sigma}$ and $\ddot{x}$ continuous across the material interface. Any *a priori* relationship that may hold between $\dot{\epsilon}$ and $\dot{\epsilon}_0$ is dependent on the constitutive equation for the materials.

Application of the jump conditions to the incident wave gives

$$(\dot{\epsilon}_1-\dot{\epsilon}_0)U_{N1}+(\ddot{x}_1-\ddot{x}_0)=0,\quad (\dot{\sigma}_1-\dot{\sigma}_0)+\rho_{01}U_{N1}(\ddot{x}_1-\ddot{x}_0)=0. \tag{A5}$$

Following collision of the incident wave with the contact surface the situation is as shown in Fig. 5. Application of the jump conditions to the reflected and transmitted waves gives

$$(\dot{\epsilon}_3-\dot{\epsilon}_1)U_{N1}-(\ddot{x}_2-\ddot{x}_1)=0,\quad (\dot{\sigma}_2-\dot{\sigma}_1)-\rho_{01}U_{N1}(\ddot{x}_2-\ddot{x}_1)=0, \tag{A6}$$

$$(\dot{\epsilon}_0-\dot{\epsilon}_1)U_{N2}-(\ddot{x}_0-\ddot{x}_2)=0,\quad (\dot{\sigma}_0-\dot{\sigma}_2)+\rho_{02}U_{N2}(\ddot{x}_0-\ddot{x}_2)=0.$$

From these equations we find that

$$\begin{aligned}\dot{\sigma}_2&=\frac{\rho_{01}U_{N1}\dot{\sigma}_0+\rho_{01}U_{N2}\dot{\sigma}_1-\rho_{01}U_{N1}\rho_{02}U_{N2}(\ddot{x}_1-\ddot{x}_0)}{\rho_{01}U_{N1}+\rho_{02}U_{N2}},\\ \ddot{x}_2&=\frac{\rho_{01}U_{N1}\ddot{x}_1+\rho_{02}U_{N2}\ddot{x}_0+\dot{\sigma}_0-\dot{\sigma}_1}{\rho_{01}U_{N1}+\rho_{02}U_{N2}},\end{aligned} \tag{A7}$$

or

$$\sigma_2=\frac{2\rho_{02}U_{N2}\dot{\sigma}_1+(\rho_{01}U_{N1}-\rho_{02}U_{N2})\dot{\sigma}_0}{\rho_{01}U_{N1}+\rho_{02}U_{N2}},$$

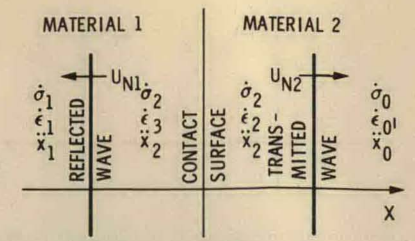

FIG. 5. Acceleration wave reflected from and transmitted through a contact surface.

$$\ddot{x}_2=\frac{2\rho_{01}U_{N1}\ddot{x}_1-(\rho_{01}U_{N1}-\rho_{02}U_{N2})\ddot{x}_0}{\rho_{01}U_{N1}+\rho_{02}U_{N2}},$$

and

$$\begin{aligned}\dot{\epsilon}_3&=\frac{2\rho_{02}U_{N2}\dot{\epsilon}_1+(\rho_{01}U_{N1}-\rho_{02}U_{N2})\dot{\epsilon}_0}{\rho_{01}U_{N1}+\rho_{02}U_{N2}},\\ \dot{\epsilon}_2&=\dot{\epsilon}_{01}+\frac{2\rho_{01}U_{N1}(U_{N1}/U_{N2})(\dot{\epsilon}_1-\dot{\epsilon}_0)}{\rho_{01}U_{N1}+\rho_{02}U_{N2}}.\end{aligned} \tag{A8}$$

For elastic materials, $U_1^2\dot{\epsilon}_0=U_2^2\dot{\epsilon}_{01}$, $U_1^2\dot{\epsilon}_3=U_2^2\dot{\epsilon}_2$.

Reflection at a free boundary

The free-surface reflection problem is a special case of the foregoing contact surface interaction problem in which the second material is replaced by a void. In this case

$$\dot{\sigma}_0=0,\quad \rho_{02}U_{N2}=0, \tag{A9}$$

and (A8) becomes

$$\dot{\sigma}_2=0,\quad \ddot{x}_2=2\ddot{x}_1-\ddot{x}_0,\quad \dot{\epsilon}_3=\dot{\epsilon}_0, \tag{A10}$$

Graphical solution of acceleration wave interaction problems

As we have seen, a plane acceleration wave of uniaxial strain in which the material passes from state 1 to state 2 must satisfy the jump condition (5.2):

$$(\dot{\sigma}_2-\dot{\sigma}_1)+\rho_0U_N(\ddot{x}_2-\ddot{x}_1)=0.$$

For a fixed state 1, this means that all accessible states 2 lie on a straight line in the $(\dot{\sigma},\ddot{x})$ plane. This line passes through $(\dot{\sigma}_1,\ddot{x}_1)$ and has slope $-\rho_0U_N$. The $(\dot{\sigma},\ddot{x})$ plane is particularly useful for considering wave interactions because the values of these quantities are continuous across contact surfaces.

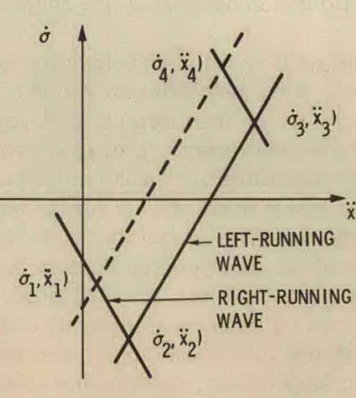

FIG. 6. Stress rate—acceleration diagram of acceleration wave collision with a contact surface.

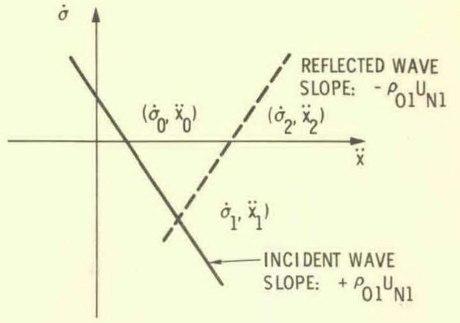

FIG. 7. Stress rate—acceleration diagram of acceleration wave reflection from a stress-free boundary.

Collision of two acceleration waves

Let us consider the interaction of a right-running wave connecting the state $(\dot\sigma_2, \ddot x_2)$ with the state $(\dot\sigma_2, \ddot x_2)$ and a left-running wave connecting the state $(\dot\sigma_2, \ddot x_2)$ with the state $(\dot\sigma_3, \ddot x_3)$. The characteristic lines for each of these waves are as shown in Fig. 6. State 2 must lie at the intersection of the left- and right-running wave characteristics, because this is the only point on both lines. This state satisfies (A1). After the wave interaction the waves adjacent to states 1 and 3 are propagating in the opposite direction to what they were previously, so the state 4 between them must lie on each of the dotted cross curves, as shown in the Fig. 6.

Collision of an acceleration wave with a contact surface

As with any acceleration wave, the incident wave can be represented by a straight line connecting states in the $(\dot\sigma, \ddot x)$ plane. The reflected wave lies on a line through $(\dot\sigma, \ddot x_1)$ with slope $\rho_{01}U_{N1}$ and the transmitted wave lies on a line through $(\dot\sigma_0, \ddot x_0)$ with slope $-\rho_{02}U_{N2}$. Since both $\dot\sigma$ and $\ddot x$ are continuous across the contact surface, the state between the two waves corresponds to the point $(\dot\sigma_2, \ddot x_2)$ where the transmitted- and reflected-wave curves intersect. This solution satisfies (A7) as, of course, it must.

Reflection of an acceleration wave from a free surface

As mentioned previously, the free-surface reflection problem is a special case of the contact surface interaction problem in which the second material is replaced by a void. In this case $\dot\sigma = 0$ and $\rho_{02}U_{N2} = 0$, and we have the situation illustrated in Fig. 7. From the drawing we obtain the results (5.10) given previously.

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